Chapter 11

Solving Systems of Equations

In This Chapter

- Finding solutions for systems of two, three, or more linear equations
- Determining if and where lines and parabolas intersect
- Expanding the search for intersections to other curves

A *system of equations* consists of a number of equations with common variables — variables that are linked in a specific way. The solution of a system of equations consists of the sets of numbers that make each equation in the system a true statement or a list of relationships between numbers that makes each equation in the system a true statement.

In this chapter, I cover both systems of linear equations and some nonlinear equations. You have a number of techniques at your disposal to solve systems of equations, including graphing lines, adding multiples of one equation to another, and substituting one equation into another.

Looking at Solutions Using the Standard Linear-Systems Form

The standard form for a system of linear equations is as follows:

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 + \ldots = k_1 \\ b_1x_1 + b_2x_2 + b_3x_3 + \ldots = k_2 \\ c_1x_1 + c_2x_2 + c_3x_3 + \ldots = k_3 \\ \vdots \end{cases}$$

The x's all represent variables, the k's are constants, and the a, b, c, and so on all represent constant coefficients of the variables.

If a system has only two linear equations with two variables, the equations appear in the $Ax + By = C$ form and can be graphed on the coordinate system to illustrate the solution. But a system of equations can contain any number of equations. (I show you how to work through larger systems in the later section, "Increasing the Number of Equations.")

Linear equations, like $Ax + By = C$, with two variables have lines as graphs. In order to solve a system of two linear equations with two variables, you need to determine what values for x and y make both the equations true at the same time. Your job is to account for which of three possible types of solutions (if you count "no solution" as a solution) can make this happen:

- **One solution:** The solution appears at the point where the lines intersect — the same x and the same y work at the same time in both equations.
- **An infinite number of solutions:** The equations are describing the same line.
- **No solution:** Occurs when the lines are parallel — no value for (x, y) works in both equations.

Solving Linear Systems by Graphing

To solve systems of linear equations with two equations and two variables (and integers as solutions), you can graph both equations on the same axes and you see one of three things: intersecting lines (one solution), identical lines (infinitely many solutions), or parallel lines (no solution).

Solving linear systems by graphing the lines created by the equations is very satisfying to your visual senses, but beware: Using this method to find a solution requires careful plotting of the lines. Also, the task of determining rational (fractions) or irrational (square roots) solutions from graphs on graph paper is too difficult, if not impossible. In general, solving systems by graphing isn't very practical.

Interpreting an intersection

Lines are made up of many, many points. When two lines cross one another, they share just one of those points. You need to graph very carefully, using a sharpened pencil and ruler with no bumps or holes.

A quick way to sketch two lines is to find their *intercepts* (where they cross the axes). Plot the intercepts on a graph and draw a line through them.

If the two lines clearly intersect at a point, you mark the point and determine the solution by counting the grid marks in the figure. This method shows you how important it is to graph the lines very carefully!

Tackling the same line

A unique situation that occurs with systems of linear equations happens when everything seems to work. Every point you find that works for one equation works for the other, too.

This match-made-in-heaven scenario plays out when the equations are just two different ways of describing the same line.

When two equations in a system of linear equations represent the same line, the equations are multiples of one another.

Putting up with parallel lines

Parallel lines never intersect and never have anything in common except their *slope.* So, when you solve systems of equations that have no solutions at all, you should know right away that the lines represented by the equations are parallel.

TIP

One way you can predict that two lines are parallel — and that no solution exists for the system of equations — is by checking the slopes of the lines. You can write each equation in *slope-intercept form* for the line. The slope-intercept form for the line $x + 2y = 8$, for example, is $y = -\frac{1}{2}x + 4$, and the slope-intercept form for $3x + 6y = 7$ is $y = -\frac{1}{2}x + \frac{7}{6}$. The lines both have the slope $-\frac{1}{2}$, and their y-intercepts are different, so you know the lines are parallel.

Using Elimination (Addition) to Solve Systems of Equations

Even though graphing lines to solve systems of equations is more visually satisfying, as a technique for solving systems of equations, graphing is time-consuming and requires careful plotting of points and cooperative answers. The two most preferred (and common) methods for solving systems of two linear equations are *elimination,* which I cover in this section, and *substitution,* which I cover in the section "Finding Substitution to Be a Satisfactory Substitute," later in this chapter. Determining which method you should use depends on what form the equations start out in and, often, personal preference.

To carry out the elimination method, you want to add two equations together, or subtract one from another other, and eliminate (get rid of) one of the variables. Sometimes you have to multiply one or both of the equations by a carefully selected number before you add them together (or subtract them).

Solve the following system of equations:

$$\begin{cases} 3x - 2y = 17 \\ 2x - 5y = 26 \end{cases}$$

The system requires some adjustments before you add or subtract the two equations. You have several different options to choose from to make the equations in this example system ready for elimination, and the one I would choose is to multiply the first equation by 2 and the second by –3 and then add to eliminate the x's.

Here's the new version of the system:

$$\begin{cases} 6x - \ \ 4y = 34 \\ -6x + 15y = -78 \end{cases}$$

Adding the two equations together, you get $11y = -44$, eliminating the x's. Dividing each side of the new equation by 11, you get $y = -4$. Substitute this value into the first *original* equation. Substituting –4 for the y value, you get $3x - 2(-4) = 17$. Solving for x, you get $x = 3$. Now check your work by putting the 3 and –4 into the second original equation. You get $2(3) - 5(-4) = 26$; $6 + 20 = 26$; $26 = 26$. Check! The solution is (3, –4).

When you graph systems of two linear equations, it becomes pretty apparent when the systems produce parallel lines or have equations that represent the same line. But you can also recognize these situations algebraically, if you know what to look for.

- When doing the algebra using elimination or substitution and you get an *impossible statement,* such as $0 = 5$, then the false statement is your signal that the system doesn't have a solution and that the lines *are parallel.*

- If the algebra results in an equation that's *always* true, such as 0 = 0 or 5 = 5, then you know that the original equations are just two ways of giving you the *same line.*

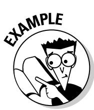
Finding Substitution to Be a Satisfactory Substitute

Another method used to solve systems of linear equations is called *substitution.* Substitution works best when solving nonlinear systems, so some people prefer sticking to substitution for both types. The method used is often just a matter of personal choice.

Variable substituting made easy

Executing substitution in systems of two linear equations is a two-step process:

1. **Solve one of the equations for one of the variables, *x* or *y*.**
2. **Substitute the value of the variable into the other equation.**



Solve the following system by substitution:

$$\begin{cases} 2x - y = 1 \\ 3x - 2y = 8 \end{cases}$$

First look for a variable that is a likely candidate for the first step. In other words, you want to solve for it.

Look for a variable with a coefficient of 1 or –1, if possible. The y term of the first equation has a coefficient of –1, so you solve this equation for y (rewrite it so y is alone on one side of the equation). You get $y = 2x - 1$. Now you can substitute the $2x - 1$ for the y in the other equation:

$$\begin{aligned} 3x - 2y &= 8 \\ 3x - 2(2x - 1) &= 8 \\ 3x - 4x + 2 &= 8 \\ -x &= 6 \\ x &= -6 \end{aligned}$$

You've already created the equation $y = 2x - 1$, so you can put the value $x = -6$ into the equation to get y:

$$y = 2(-6) - 1 = -12 - 1 = -13$$

To check your work, put both values, $x = -6$ and $y = -13$, into the equation that you didn't change (the second equation, in this case): $3(-6) - 2(-13) = 8$; $-18 + 26 = 8$; $8 = 8$. Your work checks out. Your solution is $(-6, -13)$.

Writing solutions for coexisting lines

As I mention in the section "Recognizing situations with parallel and coexisting lines," earlier in this chapter, you want to identify the impossible (parallel lines) and always possible (coexisting lines). And then, with equations that represent the same line, you can say more about a solution.

The following system of equations represents two ways of saying the same equation — two equations that represent the same line:

$$\begin{cases} 3x - 2y = 4 \\ y = \dfrac{3}{2}x - 2 \end{cases}$$

When you solve the system by using substitution, you can end up with the equation: $4 = 4$.

When substitution creates an equation that's always true, any pair of values that works for one equation will work for the other. For this reason, you can write the solution in the (x, y) form, showing a pattern or formula for all the solutions.

In the following system, the y value is always 2 less than $\frac{3}{2}$ the x value (you get this from the second equation):

$$\begin{cases} 3x - 2y = 4 \\ y = \frac{3}{2}x - 2 \end{cases}$$

So the (x, y) form for the solution of the system is $\left(x, \frac{3}{2}x - 2\right)$.

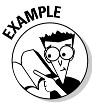
Some solutions found, using the format, are: (2, 1), $\left(3, \frac{5}{2}\right)$, and (4, 4).

Taking on Systems of Three Linear Equations

Systems of three linear equations may also have solutions: sets of numbers (all the same for each equation) that make each of the equations true. What I show you in this section, involving three equations, can be extended to four, five, or even more equations. The basic processes are the same.

Finding the solution of a system of three linear equations

When you have a system of three linear equations and three unknown variables, you solve the system by reducing the three equations with three variables into a system of two equations with two variables. At that point, you're back to familiar territory and you have all sorts of methods at your disposal to solve the system (see the previous sections in this chapter). After you determine the values of the two variables in the new system, you *back-substitute* into one of the original equations to solve for the value of the third variable.



Solve the following system:

$$\begin{cases} 3x - 2y + z = 17 \\ 2x + y + 2z = 12 \\ 4x - 3y - 3z = 6 \end{cases}$$

First, you choose a variable to eliminate. The prime two candidates for elimination are the *y* and *z* because of the coefficients of 1 or –1 that occur in their equations. Assume that you choose to eliminate the *z* variable.

Start by multiplying the terms in the top equation by –2 and adding them to the terms in the middle equation. Then, multiply the terms in the top equation (the original top equation) by 3 and add them to the terms in the bottom equation:

$$\begin{aligned} -2(3x - 2y + z = 17) \rightarrow -6x + 4y - 2z &= -34 \\ \underline{2x + 4y + 2z} &\underline{= -12} \\ -4x + 5y \qquad &= -22 \end{aligned}$$

$$\begin{aligned} 3(3x - 2y + z = 17) \rightarrow 9x - 6y + 3z &= 51 \\ \underline{4x - 3y - 3z} &\underline{= 56} \\ 13x - 9y \qquad &= 57 \end{aligned}$$

Now deal with the two equations you created by solving them as a new system of equations with just two variables. Solve it by multiplying the terms in the first equation by 9 and the terms in the second equation by 5; add the two equations together, getting rid of the *y* terms, and solving for *x*:

$$\begin{aligned} -36x + 45y &= -198 \\ \underline{65x - 45y} &\underline{= 285} \\ 29x \qquad &= 87 \\ x &= 3 \end{aligned}$$

Now you substitute $x = 3$ into the equation $-4x + 5y = -22$. Choosing this equation is just an arbitrary choice — either equation will do. When you substitute $x = 3$, you get $-4(3) + 5y = -22$. Adding 12 to each side, you get $5y = -10$, or $y = -2$.

Putting $x = 3$ and $y = -2$ into the first equation, you get $3(3) - 2(-2) + z = 17$, giving you $9 + 4 + z = 17$. You subtract 13 from each side for a result of $z = 4$. Your solution is $x = 3$, $y = -2$, $z = 4$, or you can write it as an *ordered triple,* (3, –2, 4).

Generalizing with a system solution

When dealing with three linear equations and three variables, you may come across a situation where one of the equations is a linear combination of the other two. This means you won't find a single solution for the system — but you may find an infinite number of solutions or none at all. A generalized (giving infinitely many) solution looks like $(-z, 2z, z)$, where you can pick numbers for z that determine what the x and y values are.

You first get an inkling that a system has a generalized answer when you find out that one of the reduced equations you create is a multiple of the other.

Solve the following system:

$$\begin{cases} 2x + 3y - z = 12 \\ x - 3y + 4z = -12 \\ 5x - 6y + 11z = -24 \end{cases}$$

To solve this system, you can eliminate the z's by multiplying the terms in the first equation by 4 and adding them to the second equation. You then multiply the terms in the first equation by 11 and add them to the third equation:

$$\begin{array}{rl} 4(2x + 3y - z = 12) \rightarrow & 8x + 12y - 4z = 48 \\ & \underline{x - 3y + 4z = -12} \\ & 9x + 9y \phantom{{}-4z} = 36 \end{array}$$

$$\begin{array}{rl} 11(2x + 3y - z = 12) \rightarrow & 22x + 33y - 11z = 132 \\ & \underline{5x - 6y + 11z = -24} \\ & 27x + 27y \phantom{{}-11z} = 108 \end{array}$$

The second equation, $27x + 27y = 108$, is three times the first equation. Because these equations are multiples of one another, you know that the system has infinitely many solutions — not just a single solution.

To find those solutions, you take one of the equations and solve for a variable. You may choose to solve for y in $9x + 9y = 36$. Dividing through by 9, you get $x + y = 4$. Solving for y, you get $y = 4 - x$. You substitute that equation into one of the original equations in the system to solve for z in terms of x. After you solve for z this way, you have the three variables all written as some version of x.

Substituting $y = 4 - x$ into $2x + 3y - z = 12$, for example, you get

$$\begin{aligned} 2x + 3(4 - x) - z &= 12 \\ 2x + 12 - 3x - z &= 12 \\ -x - z &= 0 \\ z &= -x \end{aligned}$$

The ordered triple giving the solutions of the system is $(x, 4 - x, -x)$. You can find an infinite number of solutions, all determined by this pattern. Just pick an x, such as $x = 3$, and then the solution is $(3, 1, -3)$. These values of x, y, and z all work in the equations of the original system.

Increasing the Number of Equations

Systems of linear equations can be any size. You can have two, three, four, or even a hundred linear equations. (After you get past three or four, you definitely need to resort to technology.) Some of these systems have solutions and others don't. You have to dive in to determine whether you can find a solution or not. You can try to solve a system of just about any number of linear equations, but you find a single, unique solution (one set of numbers for the answer) only when the number of equations isn't smaller than the number of variables. If a system has three different variables, you need at least three different equations. Having enough equations for the variables doesn't guarantee a unique solution, but you have to at least start out that way.

The general process for solving *n* equations with *n* variables is to keep eliminating variables. A systematic way is to start with the first variable, eliminate it, move to the second variable, eliminate it, and so on until you create a reduced system with two equations and two variables. You solve for the solutions of that system and then start substituting values into the original equations. This process can be long and tedious, and errors are easy to come by, but if you have to do it by hand, this is a very effective method. Technology, however, is most helpful when systems get unmanageable.

The following system has five equations and five variables:

$$\begin{cases} x + y + z + w + t = 3 \\ 2x - y + z - w + 3t = 28 \\ 3x + y - 2z + w + t = -8 \\ x - 4y + z - w + 2t = 28 \\ 2x + 3y + z - w + t = 6 \end{cases}$$

You begin the process by eliminating the x's:

1. **Multiply the terms in the first equation by –2 and add them to the second equation.**
2. **Multiply the first equation through by –3 and add the terms to the third equation.**
3. **Multiply the first equation through by –1 and add the terms to the fourth equation.**
4. **Multiply the first equation through by –2 and add the terms to the last equation.**

After you finish (whew!), you get a system with the x's eliminated:

$$\begin{cases} -3y - z - 3w + t = 22 \\ -2y - 5z - 2w - 2t = -17 \\ -5y - 2w + t = 25 \\ y - z - 3w - t = 0 \end{cases}$$

Now you eliminate the y's in the new system by multiplying the last equation by 3, 2, and 5 and adding the results to the first, second, and third equations, respectively:

$$\begin{cases} -4z - 12w - 2t = 22 \\ -7z - 8w - 4t = -17 \\ -5z - 17w - 4t = 25 \end{cases}$$

You eliminate the z's in the latest system by multiplying the terms in the first equation by 7 and the second by –4 and adding them together. You then multiply the terms in the second equation by 5 and the third by –7 and add them together. The new system you create has only two variables and two equations:

$$\begin{cases} -52w + 2t = 222 \\ 79w + 8t = -260 \end{cases}$$

To solve the two-variable system in the most convenient way, you multiply the first equation through by –4 and add the terms to the second:

$$\begin{aligned} 208w - 8t &= -888 \\ \underline{79w + 8t} &\underline{= -260} \\ 287w \qquad &= -1148 \\ w &= -4 \end{aligned}$$

You find $w = -4$. Now back-substitute w into the equation $-52w + 2t = 222$ to get $-52(-4) + 2t = 222$, which simplifies to

$$\begin{aligned} 208 + 2t &= 222 \\ 2t &= 14 \\ t &= 7 \end{aligned}$$

Take these two values and plug them into $-4z - 12w - 2t = 22$. Substituting, you get $-4z - 12(-4) -2(7) = 22$, which simplifies to

$$\begin{aligned} -4z + 34 &= 22 \\ -4z &= -12 \\ z &= 3 \end{aligned}$$

Put the three values into $y - z - 3w - t = 0$: $y - (3) - 3(-4) - 7 = 0$, or $y + 2 = 0$ and $y = -2$. Only one more to go!

Move back to the equation $x + y + z + w + t = 3$, and plug in values: $x + (-2) + 3 + (-4) + 7 = 3$, which simplifies to $x + 4 = 3$ and $x = -1$.

The solution reads: $x = -1$, $y = -2$, $z = 3$, $w = -4$, and $t = 7$ or, as an *ordered quintuple*, $(-1, -2, 3, -4, 7)$.

Intersecting Parabolas and Lines

A *parabola* is a predictable, smooth, U-shaped curve. A line is also very predictable; it goes up or down and left or right at the same rate forever and ever. If you put these two characteristics together, you can predict with a fair amount of accuracy what will happen when a line and a parabola share the same space.

When you combine the equations of a line and a parabola, you get one of three results:

- Two common solutions (intersecting in two places)
- One common solution (a line tangent to the parabola or parallel to the axis of symmetry)
- No solution at all (the line and parabola never cross)

REMEMBER

The easiest way to find the common solutions, or common sets of values, for a line and a parabola is to solve their system of equations algebraically. A graph is helpful for confirming your work and putting the problem into perspective, but solving the system by graphing usually isn't very efficient. When solving a system of equations involving a line and a parabola, most mathematicians use the substitution method.

You almost always substitute x's for the y in an equation, because you often see functions written with the y's equal to so many x's. You may have to replace x's with y's, but that's the exception. Just be flexible.

Determining if and where lines and parabolas cross paths

The graphs of a line and a parabola can cross in two places, one place, or no place at all. In terms of equations, these assertions translate to two common solutions, one solution, or no solution at all. Doesn't that fit together nicely?

Taking on two solutions

A parabola and a line may have two points in common. When using the substitution method, you first need to solve for one or the other variable.

Find the intersections of $y = 3x^2 - 4x - 1$ and $x + y = 5$.

Solve for y in the equation of the line: $y = -x + 5$. Now you substitute this equivalence of y into the first equation, set the new equation equal to 0, and factor as you do any quadratic equation:

$$\begin{aligned} y &= 3x^2 - 4x - 1 \text{ and } y = -x + 5 \\ -x + 5 &= 3x^2 - 4x - 1 \\ 0 &= 3x^2 - 3x - 6 \\ 0 &= 3(x^2 - x - 2) \\ 0 &= 3(x - 2)(x + 1) \end{aligned}$$

Setting each of the binomial factors equal to 0, you get $x = 2$ and $x = -1$. When you substitute those values into the equation $y = -x + 5$, you find that when $x = 2$, $y = 3$, and when $x = -1$, $y = 6$. The two points of intersection, therefore, are (2, 3) and (–1, 6). Figure 11-1 shows the graphs of the parabola ($y = 3x^2 - 4x - 1$), the line ($y = -x + 5$), and the two points of intersection.

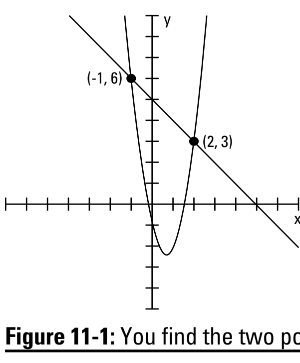


Figure 11-1: You find the two points of intersection with substitution.

Finding just one solution

When a line and a parabola have one point of intersection and, therefore, share one common solution, the line is *tangent* to the parabola or parallel to its axis of symmetry. A line and a curve can be tangent to one another if they touch or share exactly one point and if the line appears to follow the curvature at that point. (Two curves can also be tangent to one another — they touch at a point and then go their own merry ways.) The following example shows two figures that have only one point in common — at their point of tangency.

Find the intersection of $y = -x^2 + 5x + 6$ and $y = 3x + 7$.

Substitute the equivalence of y in the line equation into the parabolic equation and solve for x:

$$
\begin{aligned}
y &= -x^2 + 5x + 6 \text{ and } y = 3x + 7 \\
3x + 7 &= -x^2 + 5x + 6 \\
0 &= -x^2 + 2x - 1 \\
0 &= -1(x^2 - 2x + 1) \\
0 &= -1(x - 1)^2 \\
x &= 1
\end{aligned}
$$

The dead giveaway that the parabola and line are tangent is the quadratic equation that results from the substitution. It has a *double root* — the same solution appears twice — when the binomial factor is squared.

Substituting $x = 1$ into the equation of the line, you get $y = 3(1) + 7 = 10$. The coordinates of the point of tangency are (1, 10).

Determining that there's no solution

You can see when no solution exists in a system of equations involving a parabola and line if you graph the two figures and find that their paths never cross. But you don't need to graph the figures to discover that a parabola and line don't intersect — the algebra gives you a "no-answer answer."

Solve the system of equations containing the parabola $x = y^2 - 4y + 3$ and the line $y = 2x + 5$.

Using substitution, you get the following:

$$x = y^2 - 4y + 3 \text{ and } y = 2x + 5$$
$$x = (2x + 5)^2 - 4(2x + 5) + 3$$
$$x = 4x^2 + 20x + 25 - 8x - 20 + 3$$
$$0 = 4x^2 + 11x + 8$$

The equation looks perfectly good so far, even though the quadratic doesn't factor. You have to resort to the quadratic formula. Substituting the numbers from the quadratic equation into the formula, you get the following:

$$\begin{aligned} x &= \frac{-11 \pm \sqrt{121 - 4(4)(8)}}{2(4)} \\ &= \frac{-11 \pm \sqrt{121 - 128}}{8} \\ &= \frac{-11 \pm \sqrt{-7}}{8} \end{aligned}$$

Whoa! You can stop right there. You see that a negative value sits under the radical. The square root of –7 isn't real, so no real-number answer exists for x. (For more on non-real numbers, see Chapter 12.) The nonexistent answer is your big clue that the system of equations doesn't have a common solution, meaning that the parabola and line never intersect.

I wish I could give you an easy way to tell that a system has no solution before you go to all that work. Think of it this way: An answer of *no solution* is a perfectly good answer.

Crossing Parabolas with Circles

The graph of a parabola is a U-shaped curve, and a circle — well, you could go 'round and 'round about a circle. When a parabola and circle share some of the same coordinate plane, they can interact in one of several different ways. The two figures can intersect at four different points, three points, two points, one point, or no points at all. The possibilities may seem endless, but that's wishful thinking. The five possibilities I list here are what you have to work with. Your challenge is to determine which situation you have and to find the solutions of the system of equations. And the best way to approach this problem is algebraically.

Finding multiple intersections

A parabola and a circle can intersect at up to four different points, meaning that their equations can have up to four common solutions. The next example shows you the algebraic solution of such a system of equations.

EXAMPLE

Find the four intersections of the parabola $y = -x^2 + 6x + 8$ and the circle $x^2 + y^2 - 6x - 8y = 0$.

To solve for the common solutions, you have to solve the system of equations by either substitution or elimination. You usually don't get to use elimination in problems like this — you'd have to substitute what y is equivalent to from the parabola into the equation for the circle. It gets a bit messy. But, because I see x^2 and $-x^2$, $6x$ and $-6x$ in the two equations, I'm going to take advantage of the situation and use elimination.

First, set the equation of the parabola equal to 0, rearrange the terms in both, and add the two equations together:

$$\begin{array}{l} 0 = -x^2 + 6x \qquad\quad - y + 8 \\ \underline{0 = \quad x^2 - 6x + y^2 - 8y \quad\;} \\ 0 = \qquad\qquad\quad\;\; y^2 - 9y + 8 \end{array}$$

Now factor the quadratic for y:

$$\begin{aligned} 0 &= y^2 - 9y + 8 \\ &= (y-1)(y-8) \\ y &= 1 \text{ or } y = 8 \end{aligned}$$

Substituting 1 in for y in the equation of the parabola and solving the resulting quadratic in x, you get:

$$\begin{aligned} 1 &= -x^2 + 6x + 8 \\ x^2 - 6x - 7 &= 0 \\ (x-7)(x+1) &= 0 \\ x &= 7 \text{ or } x = -1 \end{aligned}$$

So, when $y = 1$, x is either 7 or –1. This gives you two solutions: (7, 1) and (–1, 1). You go through a similar process with $y = 8$ and get that $x = 0$ or $x = 6$. So the final two points of intersection are at (0, 8) and (6, 8).

A circle and a parabola can also intersect at three points, two points, one point, or no points.

You use the same methods to solve systems of equations that end up with fewer than four intersections. The algebra leads you to the solutions — but beware the false promises. You have to watch out for extraneous solutions by checking your answers.

If substituting one equation into another, take a look at the resulting equation. The highest power of the equation tells you what to expect as far as the number of common solutions. When the power is 3 or 4, you can have as many as three or four solutions, respectively. When the power is 2, you can have up to two common solutions. A power of 1 indicates only one possible solution. If you end up with an equation that has no solutions, you know the system has no points of intersection — the graphs just pass by like ships in the night.

Sifting through the possibilities for solutions

In the "Intersecting Parabolas and Lines" section, earlier in this chapter, the examples I provide use substitution where the x's replace the y variable. Most of the time, this is the method of choice, but I suggest you remain flexible and open for other opportunities. The next example is just such an opportunity — taking advantage of a situation where it makes more sense to replace the x term with the y term.

Find the common solutions of the parabola $y = x^2$ and the circle $x^2 + (y - 1)^2 = 9$.

Take advantage of the simplicity of the equation $y = x^2$ by replacing the x^2 in the circle equation with y. That sets you up with an equation of y's to solve:

$$\begin{aligned} x^2 + (y-1)^2 &= 9 \text{ and } y = x^2 \\ y + (y-1)^2 &= 9 \\ y + y^2 - 2y + 1 &= 9 \\ y^2 - y - 8 &= 0 \end{aligned}$$

This quadratic equation doesn't factor, so you have to use the quadratic formula to solve for y:

$$y = \frac{1 \pm \sqrt{1-4(1)(-8)}}{2(1)} = \frac{1 \pm \sqrt{33}}{2}$$

You find two different values for y, according to this solution. When you use the positive part of the $\pm$, you find that y is close to 3.37. When you use the negative part, you find that y is about –2.37. Something doesn't seem right. What is it that's bothering you? It has to be the negative value for y. The common solutions of a system should work in both equations, and $y = -2.37$ doesn't work in $y = x^2$, because when you square x, you don't get a negative number. So, only the positive part of the solution, where $y \approx 3.37$, works.

Substitute $\frac{1+\sqrt{33}}{2}$ into the equation $y = x^2$ to get x:

$$\begin{aligned} \frac{1+\sqrt{33}}{2} &= x^2 \\ \pm\sqrt{\frac{1+\sqrt{33}}{2}} &= x \end{aligned}$$

The value of x comes out to about ± 1.84. The graph in Figure 11-2 shows you the parabola, the circle, and the points of intersection at about (1.84, 3.37) and about (–1.84, 3.37).

When $y = -2.37$, you get points that lie on the circle, but these points don't fall on the parabola. The algebra shows that, and the picture agrees.

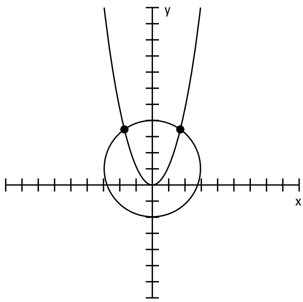


Figure 11-2: This system has only two points of intersection.

TIP

When substituting into one of the original equations to solve for the other variable, always substitute into the *simpler* equation — the one with smaller exponents. This helps you catch any extraneous solutions.